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HW II: MTH 420, Spring 2018

30/30

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QUESTION 1. Let R_1, R_2 be commutative rings, and $A = R_1 \times R_2$, with the normal operations +, . on A, i.e., $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b)(c, d) = (ac, bd)$.

(i) I will prove this in CLASS, but you may use this result for this HW. Let H be an ideal of A. Then $H = I_1 \times I_2$ for some ideal I_1 of R_1 and for some ideal I_2 of R_2.

(ii) Let F be a maximal ideal of A. Prove that $F = M \times R_2$ for some maximal ideal M of R_1 or $F = R_1 \times M$ for some maximal ideal M of R_2 (Hint: Let F be an ideal of A. Then we know that $F = I_1 \times I_2$ for some ideal I_1 of R_1 and for some ideal I_2 of R_2... now start cooking) 7

(iii) Let F be a prime ideal of A. Prove that $F = P \times R_2$ for some prime ideal P of R_1 or $F = R_1 \times P$ for some prime ideal P of R_2 5

(iv) Let F be a primary ideal of A. Prove that $F = P \times R_2$ for some primary ideal P of R_1 or $F = R_1 \times P$ for some primary ideal P of R_2 5

QUESTION 2. (i) Let R be ring and $w \in Nil(R)$. Prove that $1 + w \in U(R)$. If R is commutative, then prove that $u + w \in U(R)$ for every $u \in U(R)$ and for every $w \in N(R)$ (Hint: If n is odd, how do we factor $x^n + 1$?) 5

(ii) If R is commutative, then Prove that $N(R)$ is an ideal of R 5

(iii) If R is commutative, prove that $N(R) \subseteq P$ for every prime ideal P of R. 5

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ANSWER 1: $A = R_1 \times R_2$. $(a,b)*(c,d) = (ac, bd)$
$(a,b)+(c,d) = (a+c, b+d)$.

(i) we use the fact that If H is an Ideal of A, then $H = I_1 \times I_2$, where I_1 is an Ideal of R_1 and I_2 is an Ideal of R_2.

(ii) F is a Maximal Ideal of A. To Show: $F = M \times R_2$ (OR) $R_1 \times M$

All Maximal Ideals are Prime.

$\therefore$ Since $(0,0) \in F$, we have $(1,0)*(0,1) \in F$

Since F is prime, $(1,0) \in F$ OR $(0,1) \in F$

$$\Downarrow \qquad\qquad \Downarrow$$

$$I_1 = R_1 \qquad\qquad I_2 = R_2.$$

without loss of generality, consider the case $I_1 = R_1$.

$\therefore F = \{(r_1, i_2) \mid r_1 \in R_1 \text{ and } i_2 \in I_2\}$.

OK. Since F is proper, I_2 is proper you need this fact

To Show: I_2 is Maximal.

Since F is Maximal, $(1,1) = (r_1, i_2) + (a_1, a_2)*(c_1, c_2)$
for some $(a_1, a_2) \in A \setminus H$ and $(c_1, c_2) \in A$.

$\therefore (1,1) = (r_1 + a_1 c_1, i_2 + a_2 c_2) \Rightarrow i_2 + a_2 c_2 = 1$

Here, $i_2 \in I_2$ and $c_2 \in R_2$ and $a_2 \in R_2 \setminus I_2$

$(\because (a_1, a_2) \in A \setminus H \Rightarrow$ Either $a_1 \notin I_1$ (or) $a_2 \notin I_2$.
But $a_1 \notin I_1$ is not possible as $I_1 = R_1)$.

$\therefore I_2$ is Maximal $\Longrightarrow F = R_1 \times M$. Similarly, in Case 2:
(OR) $F = M \times R_2$

(iii) F is a prime Ideal of $A \Rightarrow F = I_1 \times I_2$.

Since $(0,0) \in F$, i.e. $(1,0) * (0,1) \in F$ and F is PRIME

$\therefore$ $(1,0) \in I_1$ (or) $(0,1) \in I_2$

$\Downarrow$ $\Downarrow$

$I_1 = R_1$ $I_2 = R_2$.

Assume $I_2 = R_2$ (Case I).

To show: I_1 is prime, i.e: $i_1 i_2 \in I_1 \longrightarrow i_1 \in I_1$ (or) $i_2 \in I_1$.

F is prime $\Rightarrow$ Whenever $(i_1, r_1) * (i_2, r_2) \in F$

Either $(i_1, r_1) \in F$ (or) $(i_2, r_2) \in F$

i.e. $(i_1 i_2, r_1 r_2) \in F \rightarrow (i_1, r_1) \in F$ (or) $(i_2, r_2) \in F$

But $F = I_1 \times R_2 \Rightarrow$ whenever $i_1 i_2 \in I_1 \rightarrow i_1 \in I_1$ (or) $i_2 \in I_2$

i I_1 is prime $\Rightarrow$ $\underline{\underline{F = P \times R_2}}$

Similarly, In other case, we get $\underline{\underline{F = R_1 \times P}}$.

(iv) F is primary $\Rightarrow F = I_1 \times I_2$.

Step I: Show $(1,0) \in F$ (or) $(0,1) \in F$.

Since $(0,0) = (1,0) * (0,1) \in F$ and F is primary

Assume $(1,0) \notin F$. Then $(0,1)^n \in F$ for some n

But $(0,1)^n = (0^n, 1^n) = (0,1)$ $\forall n$.

$\therefore$ $(1,0) \notin F \Rightarrow (0,1) \in F$

and similarly $(0,1) \notin F \Rightarrow (1,0) \in F$.

$\therefore \quad F = R_1 \times I_2 \quad (\text{OR}) \quad F = I_1 \times R_2$

Assume $F = R_1 \times I_2$.

To show: I_2 is primary, i.e. $i_1 i_2 \in I_2 \wedge i_1 \notin I_2 \Rightarrow \exists n$ s.t. $i_2^n \in I_2$

since F is primary

$(r_1, i_1) * (r_2, i_2) \in F$ and $(r_1, i_1) \notin F \Rightarrow (r_2, i_2)^n \in F$ for some n.

i.e. $(r_1 r_2, i_1 i_2) \in F$ and $(r_1, i_1) \notin F \Rightarrow (r_2, i_2)^n \in F$.

But $(r_1, i_1) \notin F$ is equivalent to $i_1 \notin I_2$.
($\because r_1 \notin R_1$ is never possible)

$\therefore$ We have: $i_1 i_2 \in I_2$ and $i_1 \notin I_2 \Rightarrow i_2^n \in I_2$ for this n.

$\therefore I_2$ is Primary (call it P) $\Rightarrow$ $F = R_1 \times P$

Similarly, In other Case: $F = P \times R_2$.

Question 2: (i) $\quad w \in Nil(R)$

To Prove: $1 + w \in U(R)$ i.e. $\exists x \in R$ s.t $(1+w)x = x(1+w) = 1$.

Proof: $w \in Nil(R) \Rightarrow \exists n$ s.t. $w^n = 0$.

Note: $w^n = 0 \Longrightarrow w^{n+k} = 0 \quad \forall k \geq 0$.

$\therefore$ If n is odd, we are okay.

If $w^{n_1} = 0$ and n_1 is even, $n := n_1 + 1$. $\therefore n$ is Always odd s.t $w^n = 0$

consider:

$$(1+w)(w^{n-1} - w^{n-2} + w^{n-3} - w^{n-4} + \dots + w^2 - w + 1)$$

$$= w^{n-1} - w^{n-2} + w^{n-3} - \dots + w^2 - w + 1$$

$$+ w^n - w^{n-1} + w^{n-2} - w^{n-3} + \dots - w^2 + w \qquad = 1 + w^n$$

$$= \underline{\underline{1}}.$$

Similarly,

$$(w^{n-1} - w^{n-2} + \dots - w + 1)(1+w) = \underline{\underline{1}}.$$

$$\therefore (1+w)^{-1} = (w^{n-1} - w^{n-2} + \dots - w + 1) \text{ and } \underline{\underline{1+w \in U(R)}}$$

Part II: To Prove: $(u+w) \in U(R)$.

$$u + w = u(1 + u^{-1}w)$$

But $u^{-1}w \in N(R)$ $\quad \Big|\; \because (u^{-1}w)^n = (u^{-1})^n w^n = 0$ (R is Commutative)

$\therefore$

$1 + u^{-1}w \in U(R)$ by part (i) of this proof

Since $U(R)$ is a group under multiplication,

$$u(1+u^{-1}w) \in U(R)$$

$$\therefore \underline{\underline{u+w \in U(R)}}.$$

(ii) If R is commutative : $N(R)$ is an Ideal of R.

Let $w_1, w_2 \in N(R)$. and $x \in R$.

To Show: $w_1 - w_2 \in N(R)$ and $w_1 x \in N(R)$

Proof: $w_1 \in N(R) \Rightarrow \exists n_1$ st. $w_1^{n_1} = 0$

$w_2 \in N(R) \Rightarrow \exists n_2$ st. $w_2^{n_2} = 0$

Let $m := n_1 + n_2$. Assume: $n_2 > n_1$

Then : $$(w_1 - w_2)^m = \sum_{k=0}^{m} (-1)^m \frac{m!}{k!(m-k)!} w_1^k w_2^{m-k}$$

But : When $0 \leq k < n_1$ we have $m-k > n_2$

$\therefore w_2^{m-k} = 0$

When $k = n_1$ we have $m - k = n_2$

$\therefore w_1^k = w_2^{m-k} = 0$

When $n_1 < k \leq m$, we have $w_1^k = 0$

$\therefore (w_1 - w_2)^m = 0 \implies w_1 - w_2 \in N(R)$

To Show: $w_1 x \in N(R)$.

Since R commutes : $(w_1 x)^{n_1} = w_1^{n_1} x^{n_1} = 0 \cdot x^{n_1} = 0$

$\therefore w_1 x \in N(R)$ $\therefore N(R)$ is an IDEAL

(iii) <u>To Prove:</u> $N(R) \subseteq P$ $\forall$ Prime Ideals 'P' of R

Let $w \in N(R)$. <u>To Show:</u> $w \in P$ $\forall$ Prime Ideals 'P'.

<u>Proof:</u> $w^n = 0$ and $0 \in P$ $\forall$ P. (P is prime Ideal)

$\therefore$ $w * w^{n-1} = 0$ and $0 \in P$

$\therefore$ $w \in P$ (or) $w^{n-1} \in P$

<u>Case I:</u> $w \in P$ $\Rightarrow$ we are done

<u>Case II:</u> $w^{n-1} \in P$

$\Rightarrow$ we can write $w^{n-1} = w * w^{n-2}$
and continue this chain, till we reach
$w^2 \in P$ which forces $w \in P$.

$\therefore$ In all cases, $w \in N(R) \longrightarrow w \in P \; \forall \; P$.

$\therefore$ $N(R) \subseteq P$ $\forall$ Prime Ideals P.

$\therefore$ $N(R) \subseteq \bigcap_{\forall i} P_i$ //.